Spherical Sliced-Wasserstein

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Optimal Transport on the Sphere

- Sphere: $S^{d-1} = \{x \in \mathbb{R}^d,\ \|x\|_2 = 1\}$
- Geodesic Distance on the sphere:

$$\forall x, y \in S^{d-1},\ d(x,y) = \arccos(\langle x, y\rangle)$$

Wasserstein distance: Let $p \geq 1$, $\mu, \nu \in \mathcal{P}_p(S^{d-1})$, then

$$W_p^p(\mu,\nu) = \inf_{\gamma\in\Pi(\mu,\nu)} \int d(x,y)^p\ \mathrm{d}\gamma(x,y).$$

- Complexity $w.r.t$ number of samples n: $O(n^3 \log n)$

Wasserstein distance on $\mathcal{P}(\mathbb{R})$:

$$\forall p \geq 1, \forall \mu,\nu \in \mathcal{P}_p(\mathbb{R}),\ W_p^p(\mu,\nu) = \int_0^1 |F_\mu^{-1}(u) - F_\nu^{-1}(u)|^p\ \mathrm{d}u$$

Sliced-Wasserstein distance:

$$\forall \mu,\nu \in \mathcal{P}_p(\mathbb{R}^d),\ SW_p^p(\mu,\nu) = \int_{S^{d-1}} W_p^p(P^\theta_\#\mu, P^\theta_\#\nu)\ \mathrm{d}\lambda(\theta),$$

where $P^\theta(x) = \langle \theta, x\rangle$ and λ is the uniform distribution on S^{d-1}.

- Complexity $w.r.t$ number of samples n: $O\big(Ln(\log n + d)\big)$ with L the number of projections $\theta_1, \dots, \theta_L$
- Use Euclidean projections: not suitable to the sphere

Contributions

- New Sliced-Wasserstein discrepancy on the sphere
- New Spherical Radon transform
- Density Estimation, Sliced-Wasserstein autoencoders

Recipe of Spherical Sliced-Wasserstein

- Projections on geodesics: great circles = intersections between S^{d-1} and 2-planes
- 2-planes modeled via the Stiefel manifold

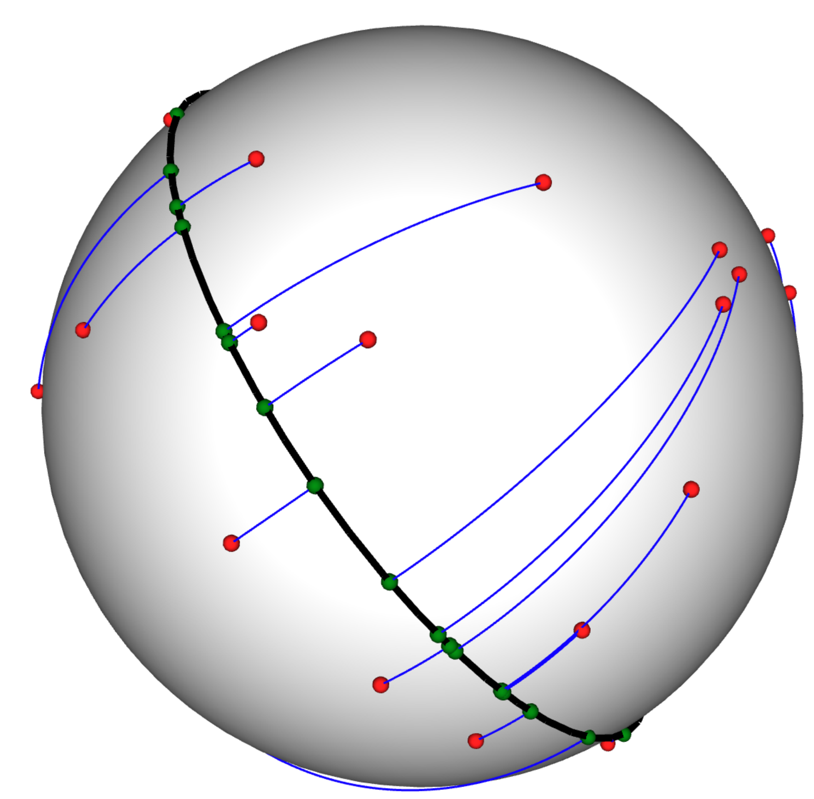
$$\mathbb{V}_{d,2} = \{U \in \mathbb{R}^{d\times 2},\ U^TU = I_2\}$$

- Geodesic projection: $P^U(x) = \mathrm{argmin}_{z\in S^1}\ d(x, Uz)$
- Wasserstein distance on the circle: $\forall \mu,\nu \in \mathcal{P}(S^1)$, (Delon et al., 2010; Rabin et al., 2011)

$$W_p^p(\mu,\nu) = \inf_{\alpha\in\mathbb{R}} \int_0^1 |F_\mu^{-1}(u) - (F_\nu - \alpha)^{-1}(u)|^p\ \mathrm{d}u,$$

$$W_2^2(\mu, \mathrm{Unif}(S^1)) = \int_0^1 |F_\mu^{-1}(t) - t - \hat{\alpha}|^2\ \mathrm{d}t,\ \hat{\alpha} = \int x\ \mathrm{d}\mu(x) - \frac{1}{2}.$$

Spherical Sliced-Wasserstein



Spherical Sliced-Wasserstein

Let $p \geq 1$, $\mu, \nu \in \mathcal{P}_p(S^{d-1})$ absolutely continuous $w.r.t.$ Lebesgue measure,

$$SSW_p^p(\mu,\nu) = \int_{\mathbb{V}_{d,2}} W_p^p(P^U_\#\mu, P^U_\#\nu)\ \mathrm{d}\sigma(U),$$

with σ the uniform distribution over $\mathbb{V}_{d,2}$.

	SW	SSW
Closed-form of W	Line	(Great)-Circle
Projection	$P^\theta(x) = \langle x, \theta\rangle$	$P^U(x) = \frac{U^Tx}{\|U^Tx\|_2}$
Integration	S^{d-1}	$\mathbb{V}_{d,2}$

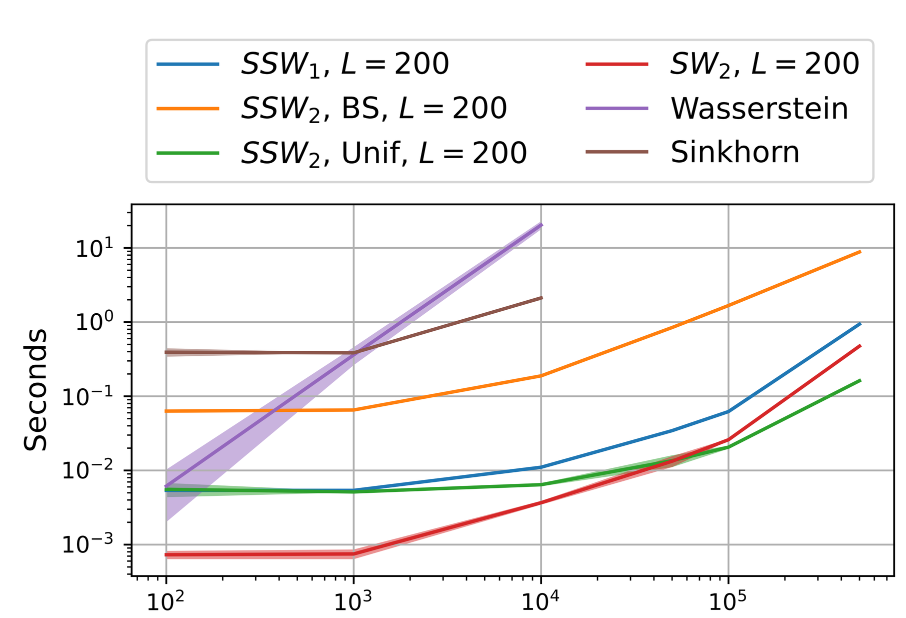
Properties

- Pseudo-distance
- Related with a Spherical Radon transform: For all $x \in S^{d-1}, z \in S^1$,

$$\tilde{R}f(z,x) = \int_{S^{d-1}} f(x) 1_{\{P^U(x)=z\}}\ \mathrm{d}\tilde{\sigma}(x).$$

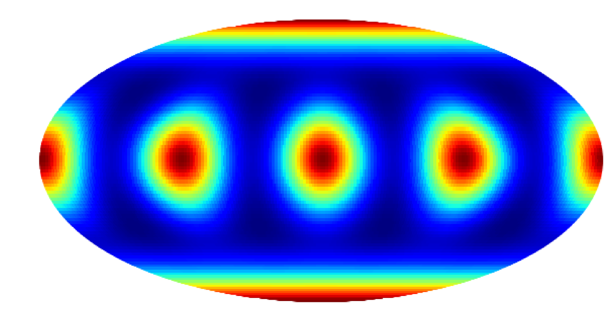
- Sample complexity independent $w.r.t.$ the dimension
- Computational complexity:

$$O\Big(Ln\big(d + \log\Big(\frac{1}{\epsilon}\Big) + \log n\big)\Big)$$

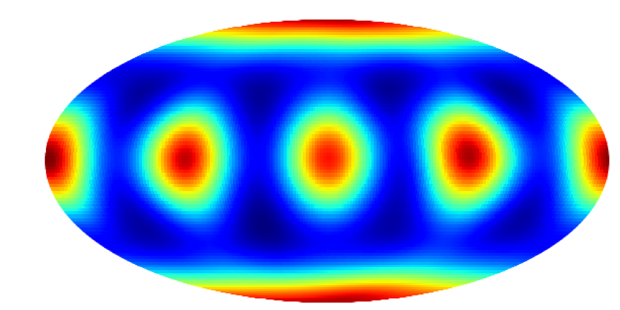


Experiments

Gradient Flow.



Target: Mixture of vMF

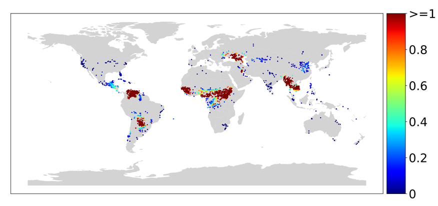


KDE estimate of 500 particles

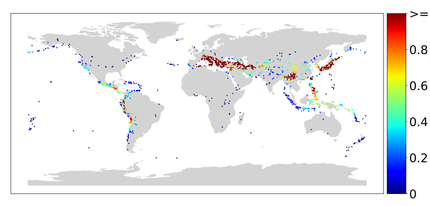
Density Estimation. Learn a normalizing flow T such that

$$\mathrm{argmin}_T\ SSW_2^2(T_\#\mu, p_Z),\ p_Z = \mathrm{Unif}(S^{d-1})$$

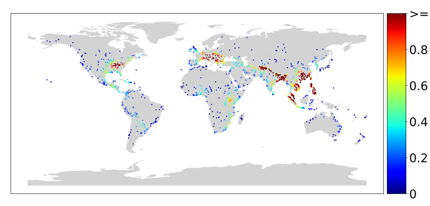
Density of μ: $\forall x \in S^{d-1},\ f_\mu(x) = p_Z\big(T(x)\big)|\det J_T(x)|$



Fire



Earthquake



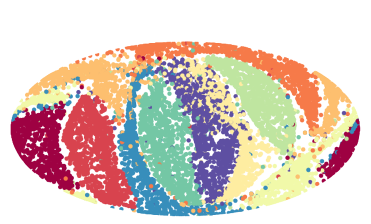
Flood

	Earthquake	Flood	Fire
SSW	**$0.84_{\pm 0.07}$**	**$1.26_{\pm 0.05}$**	**$0.23_{\pm 0.18}$**
SW	$0.94_{\pm 0.02}$	$1.36_{\pm 0.04}$	$0.54_{\pm 0.37}$
Stereo	$1.91_{\pm 0.1}$	$2.00_{\pm 0.07}$	$1.27_{\pm 0.09}$

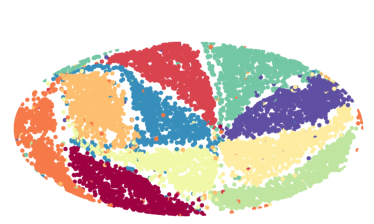
Negative test log likelihood.

SSWAE.

$$\mathcal{L}(f,g) = \int c\big(x, g(f(x))\big)\mathrm{d}\mu(x) + \lambda SSW_2^2(f_\#\mu, p_Z),$$



SWAE



SSWAE

Method / Prior	Unif(S^{10})
SSWAE	**14.91 ± 0.32**
SWAE	15.18 ± 0.32
WAE-MMD IMQ	18.12 ± 0.62
WAE-MMD RBF	20.09 ± 1.42
SAE	19.39 ± 0.56
Circular GSWAE	15.01 ± 0.26

FID (Lower is better).

References

Julie Delon, Julien Salomon, and Andrei Sobolevski. Fast transport optimization for monge costs on the circle. *SIAM Journal on Applied Mathematics*, 70(7):2239–2258, 2010.

Julien Rabin, Julie Delon, and Yann Gousseau. Transportation distances on the circle. *Journal of Mathematical Imaging and Vision*, 41(1):147–167, 2011.